Problem 1: Palindromial Polynomials - Solution

Let us take a better look at the palindromial polynomial

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 X + a_0$$

such that $a_k = a_{n-k}$ for $0 \leq k \leq n$.
If we fill in these palindromial factors of the polynomial, we see that:

$$P(X) = a_0 X^n + a_1 X^{n-1} + ... + a_1 X + a_0.$$

Now let us fill in the non-zero root $\alpha \neq \pm 1$ of P:

$$P(\alpha) = a_0 \alpha^n + a_1 \alpha^{n-1} + ... + a_1 \alpha + a_0 = 0.$$

Now, as α is non-zero, we can divide both sides of this equation by α multiple times. To be more precise, if we divide by α^n, we see that we get:

$$a_0 + a_1 \alpha^{-1} + ... + a_1 \alpha^{-(n-1)} + a_0 \alpha^{-n} = 0$$

We can rewrite this in the following way:

$$a_0 + a_1 \left(\frac{1}{\alpha}\right) + ... + a_1 \left(\frac{1}{\alpha}\right)^{n-1} + a_0 \left(\frac{1}{\alpha}\right)^n = 0$$

Now if we switch around the order, we see we get:

$$a_0 \left(\frac{1}{\alpha}\right)^n + a_1 \left(\frac{1}{\alpha}\right)^{n-1} + ... + a_1 \left(\frac{1}{\alpha}\right) + a_0 = P\left(\frac{1}{\alpha}\right) = 0.$$

Therefore we see that $\frac{1}{\alpha}$ is also a root of P. As $\alpha \neq \pm 1$, we see that this root is different from α.